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Exact Schrödinger wavefunctions of N-coupled time-dependent harmonic oscillators

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Abstract

For N-coupled time-dependent harmonic oscillators with any couplings coordinates and momentum, we have derived the primary invariant and the generalized (Lewis and Riesenfeld) invariant represented in terms of the classical solution matrix of Hamilton's oscillator equation. Using the Lewis and Riesenfeld invariant method and exploiting the properties of the symplectic matrix we have obtained the exact Schrödinger wavefunctions for the oscillators.

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1. Introduction

For a long time, much attention has been concentrated on obtaining the exact wavefunction of the Schrödinger equation for a time-dependent harmonic oscillator [1–14]. It is known that the invariant operator method is powerful in obtaining the exact wavefunction and analysing the quantum mechanical behaviours. Lewis [1] and Lewis and Riesenfeld (LR) [2] found the invariant, and derived a simple relation between eigenstates of the LR invariant and solutions of the corresponding Schrödinger equation. Using the invariant, various authors have obtained the exact wavefunctions for harmonic oscillators with time-dependent frequency [5, 12] or time-dependent mass and frequency [6, 7], linear driving terms [9]. The squeezed and coherent states were also developed from the LR invariant point of view [15, 16].

For N-coupled time-dependent harmonic oscillators which includes any couplings of coordinates and momentum, Leach [17] calculated the LR-type invariant by using the canonical transformation. Castaños *et al* [18] found the primary invariant [19], linear in position and momentum operators, by using Noether's theorem for such a system. The Ermakov–Lewis invariant was constructed for coupled linear oscillators with only coordinate–coordinate couplings [20]. Recently Ji and Hong have calculated the primary invariant for most general coupled oscillators with linear driving terms by using the Heisenberg picture approach [21].

The primary invariant are written in terms of classical solutions of Hamilton's equation for the oscillator. However, the exact wavefunctions for N-coupled time-dependent harmonic oscillators are not constructed.

In this paper, we calculate the exact Schrödinger wavefunctions of N-coupled time-dependent harmonic oscillators which includes any couplings of position and momentum, but not including linear driving terms. We first construct the primary and the LR-type invariants in terms of the classical solution matrix of Hamilton's equation using a different approach to that of [21]. It is found that the classical solution matrix is symplectic. Using the LR-type invariant and the properties of symplectic matrix we calculate the exact wavefunctions in terms of the classical solution matrix. The wavefunction constructed by Leach [17] have not been represented in terms of classical solutions of oscillators. Moreover, he only considered the two-dimensional time-dependent system. Thus our methods have an advantage on analysing the system, because the classical solutions suffice to understand the corresponding quantum system. In addition, it is found that the calculated wavefunctions are not uniquely determined. The wavefunctions depend on the choice of creation and annihilation operators.

The organization of this paper is as follows. In section 2 we construct the primary and LR-type invariants by using a different approach to the one of [21]. In section 3, using the unitary transformation and the properties of the symplectic matrix we explicitly calculate the exact wavefunctions of the Schrödinger equation in terms of eigenfunctions of the LR-type invariant. In section 4 we give simple examples. Section 5 is devoted to our summary. We use the unit $\hbar = 1$ in this paper.

2. Primary and LR-type invariants

Let us consider N-coupled time-dependent harmonic oscillators whose Hamiltonian is given by

$$H(t) = \frac{1}{2}\boldsymbol{z}^T S(t)\boldsymbol{z} \tag{1}$$

where $S(t)$ is a real symmetric $2N \times 2N$ matrix defined by four $N \times N$ matrices

$$S(t) = \begin{pmatrix} S_1 & S_2 \\ S_2^T & S_4 \end{pmatrix}. \tag{2}$$

Here we assume that $S_4(t)$ is a positive definite symmetric matrix over the time interval of interest. The equation of motion (Hamilton's equation) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{z}(t) = JS(t)\boldsymbol{z}(t) \tag{3}$$

where $\boldsymbol{z}$ is a $2N$ vector, given by $\boldsymbol{z} = \binom{q}{p}$ with $\boldsymbol{q} = (q_1, \ldots, q_N)^T$ and $\boldsymbol{p} = (p_1, \ldots, p_N)^T$. T represents the transposition. The unit symplectic matrix J is defined by

$$J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix} \tag{4}$$

where I_N is an $N \times N$ identity matrix.

The primary invariant of the form $\boldsymbol{b}(t) = V(t)\boldsymbol{z}(t)$ which satisfies the invariant equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{b}(t) = \frac{\partial}{\partial t}\boldsymbol{b}(t) - \mathrm{i}[\boldsymbol{b}(t), H(t)] = 0 \tag{5}$$

was derived by Ji and Hong [21]. Here $\boldsymbol{b} = (b_1, \ldots, b_{2N})^T$ and $V(t)$ is a $2N \times 2N$ matrix. They inserted $\boldsymbol{b}(t) = V(t)\boldsymbol{z}(t)$ into the invariant equation (5) and found that $V(t)$ satisfies the classical equation of motion (Hamilton's equation) for equation (1):

$$\dot{V} = -VJS(t) \tag{6}$$

which is the transposition of equation (3).

In this paper we take a different method to find the primary invariant. First, it is important to note that $\boldsymbol{z}(t)$ can be written as

$$\boldsymbol{z}(t) = W(t)\boldsymbol{z}(t_0) \tag{7}$$

where the matrix $W(t)$ (which is a real matrix) obeys the same equation of motion as those of $\boldsymbol{z}(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t) = JS(t)W(t) \tag{8}$$

with the initial condition $W(t_0) = I_{2N}$. $W(t)$ is just the $2N \times 2N$ fundamental solution matrix of equation (3). It is known that the matrix $W(t)$ is symplectic for all t, i.e. it satisfies $W^T JW = J$ [23]. Second, it is noted that the invariant equation (5) implies $\boldsymbol{b}(t) = \boldsymbol{b}(t_0)$, constant operators. If we require that (i) $b_i = \hat{b}_i$, $b_{N+i} = \hat{b}_i^\dagger$ and (ii) $[\hat{b}_i, \hat{b}_j] = 0$, $[\hat{b}_i, \hat{b}_j^\dagger] = \delta_{ij}$, and $[\hat{b}_i^\dagger, \hat{b}_j^\dagger] = 0$. For $i, j = 1, 2, \ldots, N$, we can write

$$\boldsymbol{b}(t) = \begin{pmatrix} \hat{\boldsymbol{b}}(t) \\ \hat{\boldsymbol{b}}^\dagger(t) \end{pmatrix} = \sqrt{\tfrac{1}{2}} \begin{pmatrix} I_N & \mathrm{i}I_N \\ I_N & -\mathrm{i}I_N \end{pmatrix} \Gamma \begin{pmatrix} \boldsymbol{q}(t_0) \\ \boldsymbol{p}(t_0) \end{pmatrix} \equiv \Omega_0 \Gamma \boldsymbol{z}(t_0) \tag{9}$$

where $\hat{\boldsymbol{b}} = (\hat{b}_1, \ldots, \hat{b}_N)^T$. The matrix Γ is a constant nonsingular symplectic matrix, i.e. $\Gamma^T J \Gamma = J$. The equations (7) and (9) mean that $\boldsymbol{b}(t) = V(t)W(t)\Gamma^{-1}\Omega_0^{-1}\boldsymbol{b}(t)$. Therefore we obtain that

$$V(t) = \Omega_0 \Gamma W^{-1}(t). \tag{10}$$

Using the property of symplectic matrix $W^{-1}(t) = -JW^T(t)J$ [23], equation (10) can be written as

$$V(t) = -\Omega_0 \Gamma J W^T(t) J. \tag{11}$$

It is trivial to show that the matrix $V(t)$ is a complex symplectic matrix, i.e. it satisfies $V^T J V = -\mathrm{i}J$.

As was discussed in [21], the solution matrix $V(t)$ is not determined uniquely because the linear combinations of $\hat{b}_i, \hat{b}_i^\dagger$ can also form other annihilation and creation operators. In other words, if we multiply a non-singular constant matrix Ω, satisfying $\Omega J \Omega^T = -\mathrm{i}J$, on equation (9) then $\tilde{\boldsymbol{b}} = \Omega V \boldsymbol{z}$ also forms a primary invariant. But ΩV can be always written in standard form (10) because $\Omega V = \Omega_0(\Omega_0^{-1}\Omega\Omega_0\Gamma)W^{-1} \equiv \Omega_0\Gamma' W^{-1}$ and $\Gamma'^T J \Gamma' = J$. Thus we have the freedom to choose $W(t)$, corresponding to the change $W(t) \to W(t)\Gamma^{-1}$. If we choose that $\Gamma = \mathrm{diag}\,(\sqrt{\omega_1}, \ldots, \sqrt{\omega_N}, 1/\sqrt{\omega_1}, \ldots, 1/\sqrt{\omega_N})$, then $V(t)$ is the same matrix as the one in [21]. The non-uniqueness of the primary invariant results in the non-uniqueness of the wavefunctions.

Finally, we have constructed the primary invariant in terms of classical solution matrix (fundamental solution matrix) of (3):

$$\boldsymbol{b}(t) = \Omega_0 \Gamma W^{-1}(t)\boldsymbol{z}(t) \tag{12}$$

which is one of the our main results.

With the primary invariants, the LR-type invariant is constructed as

$$I = \sum_{i=1}^{N} \left(\hat{b}_i^\dagger \hat{b}_i + \tfrac{1}{2}\right) \tag{13}$$

or, in terms of $\boldsymbol{z}$, as

$$I = \tfrac{1}{2}\boldsymbol{z}^T(t)V^T(t)\begin{pmatrix} 0 & I_N \\ I_N & 0 \end{pmatrix} V(t)\boldsymbol{z}(t) = \tfrac{1}{2}\boldsymbol{z}^T(t)W^{-T}\Gamma^T\Gamma W^{-1}\boldsymbol{z}(t) \tag{14}$$

where we used $\Omega_0^T \begin{pmatrix} 0 & I_N \\ I_N & 0 \end{pmatrix} \Omega_0 = I_{2N}$. If we decompose the matrix $W(t)\Gamma^{-1}$ into four $N \times N$ matrices

$$W(t)\Gamma^{-1} = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix} \tag{15}$$

the LR-type invariant $I(t)$ is represented in terms of $\boldsymbol{q}$ and $\boldsymbol{p}$ as

$$I(t) = \tfrac{1}{2}\left[\boldsymbol{p}^T C\boldsymbol{p} + \boldsymbol{p}^T B^T \boldsymbol{q} + \boldsymbol{q}^T B\boldsymbol{p} + \boldsymbol{q}^T A\boldsymbol{q}\right] \tag{16}$$

where the coefficient matrices are given by

$$A = W_4 W_4^T + W_3 W_3^T \qquad B = -W_4 W_2^T - W_3 W_1^T \qquad \text{and} \qquad C = W_1 W_1^T + W_2 W_2^T. \tag{17}$$

It is noted that the matrices A and C are positive definite. The matrices A, B and C obey the following matrix equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} = -\begin{pmatrix} S_2 A - S_1 B^T + A S_2^T - B S_1 & S_2 B - S_1 C + A S_4 - B S_2 \\ S_4 A - S_2^T B^T + B^T S_2^T - C S_1 & S_4 B - S_2^T C + B^T S_4 - C S_2 \end{pmatrix}. \tag{18}$$

As an example, we consider the $N = 1$ case. If we define as $S_4(t) = 1/M(t)$, $S_1(t) = M(t)\omega^2(t)$, $A(t) = g_+(t)$, $B(t) = g_0(t)$, and $C(t) = g_-(t)$, then equation (18) is reduced to equation (2.3) in [6] when $S_2(t) = 0$ and reduced to equation (2.6) in [9] with $S_2(t) = Y(t)$.

3. The eigenfunction of LR-type invariant and the exact Schrödinger wavefunction

In this section we calculate the exact Schrödinger wavefunctions by using the LR invariant method. For a review of the invariant method, see [22]. According to the LR [2], the wavefunction $\Psi_n(\boldsymbol{q}, t)$ of the Schrödinger equation $\mathrm{i}\partial/\partial t \Psi_n(\boldsymbol{q}, t) = H(t)\Psi_n(\boldsymbol{q}, t)$ can be written as

$$\Psi_n(\boldsymbol{q}, t) = \mathrm{e}^{\mathrm{i}\alpha_n(t)}\phi_n(\boldsymbol{q}, t) \tag{19}$$

where $\phi_n(\boldsymbol{q}, t)$ is the eigenfunction of $I(t)$. The eigenfunctions are assumed to form a complete orthonormal set with time-independent eigenvalues λ_n. Thus

$$I(t)\phi_n(\boldsymbol{q}, t) = \lambda_n \phi_n(\boldsymbol{q}, t) \tag{20}$$

with $\langle \phi_{n'} | \phi_n \rangle = \delta_{n',n}$ and $\boldsymbol{n}$ is a quantum number to be determined. The phase functions $\alpha_n(t)$ are determined by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha_n(t) = \langle \phi_n | \mathrm{i}\frac{\partial}{\partial t} - H(t) | \phi_n \rangle. \tag{21}$$

The general solution can then be written as

$$\Psi(\boldsymbol{q}, t) = \sum_{\boldsymbol{n}} C_n \mathrm{e}^{\mathrm{i}\alpha_n(t)} \phi_n(\boldsymbol{q}, t) \tag{22}$$

where C_n are time independent.

To obtain eigenfunctions $\phi_n(\boldsymbol{q}, t)$ we first perform the unitary transformations:

$$\phi_n(\boldsymbol{q}, t) = U\phi_n'(\boldsymbol{q}, t) \tag{23}$$

with the unitary operator

$$U = \mathrm{e}^{-\mathrm{i}\frac{1}{2}\boldsymbol{q}^T C^{-1} B^T \boldsymbol{q}}. \tag{24}$$

The unitary operator has the properties

$$U^{-1}\boldsymbol{p}U = \boldsymbol{p} - C^{-1}B^T\boldsymbol{q} \qquad \text{and} \qquad U^{-1}\boldsymbol{q}U = \boldsymbol{q}. \tag{25}$$

Under this unitary transformation the operator I is changed into

$$I' = U^{-1} I U \tag{26}$$

and the eigenvalue equation (20) is mapped into

$$I' \phi'_{\boldsymbol{n}}(\boldsymbol{q}, t) = \lambda_{\boldsymbol{n}} \phi'_{\boldsymbol{n}}(\boldsymbol{q}, t) \tag{27}$$

where

$$I' = \tfrac{1}{2}\big[\boldsymbol{p}^T C \boldsymbol{p} + \boldsymbol{q}^T (A - BC^{-1}B^T)\boldsymbol{q}\big]. \tag{28}$$

Using the relation (see the proof in the appendix)

$$A - BC^{-1}B^T = C^{-1} \tag{29}$$

we can write equation (28) as

$$I' = \tfrac{1}{2}\big(\boldsymbol{p}^T C \boldsymbol{p} + \boldsymbol{q}^T C^{-1} \boldsymbol{q}\big). \tag{30}$$

Now we define the new independent variable $\boldsymbol{q}'$ as $\boldsymbol{q}' = X\boldsymbol{q}$, where X is a nonsingular matrix diagonalizing the matrices C and S_4 simultaneously as

$$XCX^T = I_N \qquad \text{and} \qquad XS_4X^T = \mathrm{diag}(d_1, \ldots, d_N). \tag{31}$$

This diagonalization is always possible because C and S_4 are positive definite and symmetric matrices [24]. With this new variable $\boldsymbol{q}'$, we can write the eigenvalue equation as

$$\sum_i \left(-\frac{\partial^2}{\partial q_i'^2} + q_i'^2 \right) \varphi_{\boldsymbol{n}}(\boldsymbol{q}') = \sum_i \left(n_i + \frac{1}{2} \right) \varphi_{\boldsymbol{n}}(\boldsymbol{q}') \tag{32}$$

where

$$\phi'_{\boldsymbol{n}}(\boldsymbol{q}, t) = \sqrt{\det(X)}\, \varphi_{\boldsymbol{n}}(X\boldsymbol{q}). \tag{33}$$

Here we have taken as $\lambda_{\boldsymbol{n}} = \sum_i (n_i + \frac{1}{2})$ where $n_i = 0, 1, 2, \ldots$ for $i = 1, \ldots, N$. The quantum number $\boldsymbol{n}$ is thus $\boldsymbol{n} = (n_1, \ldots, n_N)$. The factor $\sqrt{\det(X)}$ is introduced into equation (33) so that the normalization condition

$$\int \phi'^*_{\boldsymbol{n}}(\boldsymbol{q}, t) \phi'_{\boldsymbol{n}}(\boldsymbol{q}, t)\, \mathrm{d}\boldsymbol{q} = \int \varphi^*_{\boldsymbol{n}}(\boldsymbol{q}', t) \varphi_{\boldsymbol{n}}(\boldsymbol{q}', t)\, \mathrm{d}\boldsymbol{q}' = 1 \tag{34}$$

holds. Now the transformed eigenvalue equation (32) is the Schrödinger equation for N-uncoupled time-independent harmonic oscillators. The normalized solutions of (32) are given by

$$\varphi_{\boldsymbol{n}}(\boldsymbol{q}') = \prod_i \left[\frac{1}{\pi^{1/2} n_i! 2^{n_i}} \right]^{1/2} \mathrm{e}^{-\frac{1}{2} q_i'^2} H_{n_i}\left(q'_i\right) \tag{35}$$

where H_n is the Hermite polynomial of order n. Hence we obtain the normalized eigenfunction of the invariant I as

$$\phi_{\boldsymbol{n}}(\boldsymbol{q}, t) = \sqrt{\det(X)} \mathrm{e}^{-\mathrm{i}\frac{1}{2} \boldsymbol{q}^T C^{-1} B^T \boldsymbol{q}} \prod_i \left[\frac{1}{\pi^{1/2} n_i! 2^{n_i}} \right]^{1/2} \mathrm{e}^{-\frac{1}{2} q_i'^2} H_{n_i}\left(q'_i\right) \tag{36}$$

with the eigenvalue $\lambda_{\boldsymbol{n}} = \sum_i (n_i + \frac{1}{2})$. Using $\prod_i \mathrm{e}^{-\frac{1}{2} q_i'^2} = \mathrm{e}^{-\frac{1}{2} \boldsymbol{q}^T X^T X \boldsymbol{q}} = \mathrm{e}^{-\frac{1}{2} \boldsymbol{q}^T C^{-1} \boldsymbol{q}}$, it is written as

$$\phi_{\boldsymbol{n}}(\boldsymbol{q}, t) = \sqrt{\det(X)} \mathrm{e}^{-\frac{1}{2} \boldsymbol{q}^T [\mathrm{i} C^{-1} B^T + C^{-1}] \boldsymbol{q}} \prod_i \left[\frac{1}{\pi^{1/2} n_i! 2^{n_i}} \right]^{1/2} H_{n_i}\left(q'_i\right). \tag{37}$$

The remaining problem is to obtain the phase $\alpha_n(t)$. Substituting $\phi_n(\boldsymbol{q},t)$ into equation (21), after a little algebra, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha_n(t) = -\frac{1}{2}\int \mathrm{d}\boldsymbol{q}\,\phi_n^*(\boldsymbol{q},t)\phi_n(\boldsymbol{q},t)\left\{\mathrm{Tr}\left(S_4R - \mathrm{i}S_2 - \mathrm{i}X^{-1}\dot{X}\right)\right.$$
$$+\boldsymbol{q}^T\left(R^TS_4R - \mathrm{i}R^TS_2^T - \mathrm{i}S_2R - S_1 - \mathrm{i}\dot{R}\right)\boldsymbol{q}$$
$$\left.+\sum_i\left[2\left((XS_4R - \mathrm{i}XS_2^T - \mathrm{i}\dot{X})\boldsymbol{q}\right)_i H'_{n_i}/H_{n_i} - d_iH''_{n_i}/H_{n_i}\right]\right\} \tag{38}$$

where $R = \mathrm{i}C^{-1}B^T + C^{-1}$ and we have used that XS_4X^T is a diagonal matrix. Here the prime $'$ denotes differentiation with respect to q'_i. Using the relations $C^{-1}B^T = BC^{-1}$ (see the appendix), $\dot{C}^{-1} = -C^{-1}\dot{C}C^{-1}$, equation (18), and equation (29) we can show that

$$\mathrm{i}\dot{R} = R^TS_4R - \mathrm{i}R^TS_2^T - \mathrm{i}S_2R - S_1. \tag{39}$$

And also, we can show that

$$\int \mathrm{d}\boldsymbol{q}\,\phi_n^*(\boldsymbol{q},t)\phi_n(\boldsymbol{q},t)\left[\sum_i\left((XS_4R - \mathrm{i}XS_2^T - \mathrm{i}\dot{X})\boldsymbol{q}\right)_i H'_{n_i}/H_{n_i}\right]$$
$$= \int \mathrm{d}\boldsymbol{q}\,\phi_n^*(\boldsymbol{q},t)\phi_n(\boldsymbol{q},t)\sum_i d_iq'_iH'_{n_i}/H_{n_i}. \tag{40}$$

Then equation (38) is written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha_n(t) = -\frac{1}{2}\int \mathrm{d}\boldsymbol{q}\,\phi_n^*(\boldsymbol{q},t)\phi_n(\boldsymbol{q},t)\left[\mathrm{Tr}\left(S_4R - \mathrm{i}S_2 - \mathrm{i}X^{-1}\dot{X}\right)\right.$$
$$\left.+\sum_i\left(-d_iH''_{n_i} + 2d_iq'_iH'_{n_i}\right)/H_{n_i}\right]. \tag{41}$$

Using the equation for the Hermite polynomial, $H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0$, we can obtain the simple form for the phase as

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha_n(t) = -\sum_i d_in_i - \frac{1}{2}\mathrm{Tr}(S_4R - \mathrm{i}S_2 - \mathrm{i}X^{-1}\dot{X}). \tag{42}$$

Using $\mathrm{Tr}(X^{-1}\dot{X}) = \mathrm{Tr}(S_4BC^{-1} - S_2^T)$ and $R = \mathrm{i}C^{-1}B^T + C^{-1}$, the phase function $\alpha_n(t)$ then can be expressed as

$$\alpha_n(t) = -\int^t \mathrm{d}t'\left[\sum_i d_in_i - \tfrac{1}{2}\mathrm{Tr}(S_4X^TX)\right]$$
$$= -\int^t \mathrm{d}t'\sum_i d_i(t')\left(n_i + \tfrac{1}{2}\right). \tag{43}$$

Finally, after using $\det(X) = 1/\sqrt{\det(C)}$, we obtain the exact wavefunction of the Schrödinger equation as

$$\Psi_n(\boldsymbol{q},t) = \frac{1}{C^{1/4}}\mathrm{e}^{-\mathrm{i}\int^t \mathrm{d}t'\sum_i d_i(t')(n_i+\frac{1}{2})}\mathrm{e}^{-\frac{1}{2}\boldsymbol{q}^T(\mathrm{i}C^{-1}B^T+C^{-1})\boldsymbol{q}}\prod_i\left[\frac{1}{\pi^{1/2}n_i!2^{n_i}}\right]^{1/2}H_{n_i}\left(q'_i\right) \tag{44}$$

which is one of the our main results.

4. Examples

4.1. One-dimensional time-dependent harmonic oscillator

For the $N = 1$ case, the time-dependent generalized harmonic oscillator is written as

$$H(t) = \frac{1}{2}\left[\frac{1}{M(t)}p^2 + Y(t)(qp + pq) + M(t)\omega^2(t)q^2\right] \tag{45}$$

where we defined as $S_4(t) = 1/M(t)$, $S_2(t) = Y(t)$ and $S_1(t) = M(t)\omega^2(t)$. The functions $M(t)$, $Y(t)$ and $\omega(t)$ depend on time explicitly. Let the fundamental solution matrix $W(t)$ be $\begin{pmatrix} f_1(t) & f_2(t) \\ f_3(t) & f_4(t) \end{pmatrix}$, where f_i are solutions of equation of motion for $H(t)$:

$$M\ddot{f}_i + \dot{M}\dot{f}_i + \left(M\omega^2 - MY^2 - \dot{M}Y - M\dot{Y}\right)f_i = 0 \tag{46}$$

with the initial conditions $f_1(t_0) = 1$, $f_2(t_0) = 0$, $f_3(t_0) = 0$ and $f_4(t_0) = 1$. It is well known that the general form of the 2×2 matrix Γ^{-1} is given by $\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$ with $c_1c_4 - c_2c_3 = 1$ [23]. The general form of $W\Gamma^{-1}$ is then

$$W\Gamma^{-1} = \begin{pmatrix} c_1 f_1(t) + c_3 f_2(t) & c_2 f_1(t) + c_4 f_2(t) \\ c_1 f_3(t) + c_3 f_4(t) & c_2 f_3(t) + c_4 f_4(t) \end{pmatrix}. \tag{47}$$

Then C, X, B and $d(t)$ are given by

$$C(t) = (c_1^2 + c_2^2)f_1^2 + (c_3^2 + c_4^2)f_2^2 + 2(c_1c_3 + c_2c_4)f_1f_2 \equiv \rho^2(t) \tag{48}$$

$$X(t) = \frac{1}{\rho(t)} \tag{49}$$

$$B(t) = -M(t)\dot{\rho}(t)\rho(t) + M(t)Y(t)\rho^2(t) \tag{50}$$

$$d(t) = \frac{1}{M(t)\rho^2(t)}. \tag{51}$$

The wavefunction is then expressed as

$$\Psi_n(q,t) = \left[\frac{1}{\pi^{1/2}n!2^n\rho}\right]^{1/2} \mathrm{e}^{-\mathrm{i}\int^t \mathrm{d}t' \frac{1}{M(t')\rho(t')}(n+\frac{1}{2})} \mathrm{e}^{-\frac{1}{2}[-\mathrm{i}M(t)(\frac{\dot{\rho}}{\rho}-Y)+\frac{1}{\rho^2}]q^2} H_n\left(\frac{q}{\rho}\right). \tag{52}$$

In the case of a harmonic oscillator with time-independent mass $M(t) = m$, frequency $S_1(t) = \omega_0$ and $Y(t) = 0$, we can set $f_1(t) = \cos(\omega_0 t)$ and $f_2(t) = \frac{1}{\omega_0}\sin(\omega_0 t)$. If we choose that $c_1 = \frac{1}{\sqrt{\omega_0}}$, $c_2 = 0$, $c_3 = 0$, and $c_4 = \sqrt{\omega_0}$, then the wavefunction becomes the usual stationary wavefunction. When the mass is constant, i.e. $M(t) = M_0$ and $Y(t) = 0$, the wavefunction (52) gives the one of [5] with setting $c_1 f_1(t) + c_3 f_2(t) = \rho(t)\cos(\gamma(t))$ and $c_2 f_1(t) + c_4 f_2(t) = \rho(t)\sin(\gamma(t))$, where $\cos(\gamma(t)) = [c_1 f_1(t) + c_3 f_2(t)]/\rho(t)$ and $\sin(\gamma(t)) = [c_2 f_1(t) + c_4 f_2(t)]/\rho(t)$. And the wavefunction agrees with that of [7], and that of [6] by setting $\rho^2(t) = g_-(t)/\omega_I$, and that of [9] when $Y(t) \neq 0$.

It is noted that the different choice of c_1, c_2, c_3, c_4 may give a different set of wavefunctions $\{\Psi_n, n = 1, 2, \ldots\}$, as was discussed in [10]. In order words, the wavefunctions are not uniquely determined and depend on the choice of the classical solutions. From equation (12) we can see that choosing different c_1, c_2, c_3, c_4 might amount to choosing the different annihilation and creation operators. This was suggested in [25] by investigating the Gaussian pure states. But we can fix these constants according to initial conditions of the system.

4.2. The two-dimensional uncoupled time-dependent harmonic oscillator

Let us consider the two-dimensional uncoupled anisotropic oscillator whose Hamiltonian, considered by Leach [17], is given by

$$H(t) = \tfrac{1}{2}\left[p_1^2 + \omega_1^2(t)q_1^2 + p_2^2 + \omega_2^2(t)q_2^2\right]. \tag{53}$$

Because this system is an uncoupled one, the matrices $W(t)$ and Γ may be written as

$$W(t) = \begin{pmatrix} f_1(t) & 0 & f_2(t) & 0 \\ 0 & g_1(t) & 0 & g_2(t) \\ f_3(t) & 0 & f_4(t) & 0 \\ 0 & g_3(t) & 0 & g_4(t) \end{pmatrix} \qquad \Gamma^{-1} = \begin{pmatrix} c_1(t) & 0 & c_2(t) & 0 \\ 0 & \bar{c}_1(t) & 0 & \bar{c}_2(t) \\ c_3(t) & 0 & c_4(t) & 0 \\ 0 & \bar{c}_3(t) & 0 & \bar{c}_4(t) \end{pmatrix} \tag{54}$$

with appropriate initial conditions for $f_{1,2}$ and $g_{1,2}$. Then it follows that

$$C(t) = \mathrm{diag}(\rho_1^2(t), \rho_2^2(t)) \tag{55}$$

$$X(t) = \mathrm{diag}(1/\rho_1(t), 1/\rho_2(t)) \tag{56}$$

$$B(t) = \mathrm{diag}(-\dot{\rho}_1(t)\rho_1(t), -\dot{\rho}_2(t)\rho_2(t)) \tag{57}$$

$$d_1(t) = 1/\rho_1^2(t), d_2(t) = 1/\rho_2^2(t) \tag{58}$$

where $\rho_1^2(t) = (c_1^2 + c_2^2)f_1^2 + (c_3^2 + c_4^2)f_2^2 + 2(c_1c_3 + c_2c_4)f_1f_2$ and $\rho_2^2(t) = (\bar{c}_1^2 + \bar{c}_2^2)g_1^2 + (\bar{c}_3^2 + \bar{c}_4^2)g_2^2 + 2(\bar{c}_1\bar{c}_3 + \bar{c}_2\bar{c}_4)g_1g_2$. The wavefunction is then reduced to the product of one-dimensional wavefunctions as is well known:

$$\Psi_{n_1,n_2}(q_1, q_2, t) = \Psi_{n_1}(q_1, t)\Psi_{n_2}(q_2, t) \tag{59}$$

where $\Psi_{n_1}(q_1, t)$ and $\Psi_{n_2}(q_2, t)$ are of the form of equation (52) with $M(t) = 1$ and $Y(t) = 0$. If we choose different Γ, the wavefunction is not then written in the form of the simple product of one-dimensional ones.

4.3. Generalized two-dimensional time-dependent harmonic oscillators

Let us consider a system described by the following Hamiltonian:

$$H = \frac{1}{2m(t)}\left(p_1^2 + p_2^2\right) + \frac{1}{2}m(t)\omega^2(t)\left(x_1^2 + x_2^2\right) + \lambda(t)\left(x_1p_2 - x_2p_1\right). \tag{60}$$

If $m(t)$ is a constant, the Hamiltonian (60) describes a charged particle moving in an axially symmetric time-dependent magnetic field [2]. It was also investigated to study the statistical properties of a charged oscillator in the presence of a uniform magnetic field [26]. When the frequency is constant, i.e. $\omega(t) = \omega_0$, Hamiltonian (60) was studied from the point of view of Noether's theorem [18].

The equations of motion of (60) in matrix form are written as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} = \begin{pmatrix} 0 & -\lambda & 1/m & 0 \\ \lambda & 0 & 0 & 1/m \\ -m\omega^2 & 0 & 0 & -\lambda \\ 0 & -m\omega^2 & \lambda & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{pmatrix}. \tag{61}$$

Introducing a rotation about the x_3-axis

$$\mathcal{R} = \begin{pmatrix} \cos(\alpha(t)) & -\sin(\alpha(t)) \\ \sin(\alpha(t)) & \cos(\alpha(t)) \end{pmatrix} \tag{62}$$

with time-dependent angle $\alpha(t) = \int_{t_0}^{t} \lambda(t')\,\mathrm{d}t'$ and new variables

$$\boldsymbol{q} = \mathcal{R}\boldsymbol{Q} \qquad \text{and} \qquad \boldsymbol{p} = \mathcal{R}\boldsymbol{P} \tag{63}$$

where $\boldsymbol{q} = (x_1, x_2)^T, \boldsymbol{Q} = (X_1, X_2)^T, \boldsymbol{p} = (p_1, p_2)^T$, and $\boldsymbol{P} = (P_1, P_2)^T$, we obtain new equations of motion:

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{P}_1 \\ \dot{P}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1/m & 0 \\ 0 & 0 & 0 & 1/m \\ -m\omega^2 & 0 & 0 & 0 \\ 0 & -m\omega^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ P_1 \\ P_2 \end{pmatrix} \tag{64}$$

which are equations of motion for uncoupled time-dependent harmonic oscillators. The fundamental matrices are then written as

$$W_1(t) = \mathcal{R} \begin{pmatrix} f_1 & 0 \\ 0 & g_1 \end{pmatrix} \qquad \text{and} \qquad W_2(t) = \mathcal{R} \begin{pmatrix} f_2 & 0 \\ 0 & g_2 \end{pmatrix}. \tag{65}$$

W_3 and W_4 are given by $W_3 = S_4^{-1}(\dot{W}_1 - S_2^T W_1)$ and $W_4 = S_4^{-1}(\dot{W}_2 - S_2^T W_2)$. Here f_1, f_2, g_1, and g_2 are solutions of the following equation:

$$\ddot{f} + \frac{\dot{m}}{m}\dot{f} + \omega^2 f = 0 \tag{66}$$

with appropriate initial conditions for $f_{1,2}$ and $g_{1,2}$. Introducing the matrix Γ of the form (54), after a little algebra, we obtain

$$C = \mathcal{R} \begin{pmatrix} \rho_1^2 & 0 \\ 0 & \rho_2^2 \end{pmatrix} \mathcal{R}^T \tag{67}$$

$$X = \begin{pmatrix} \frac{1}{\rho_1} & 0 \\ 0 & \frac{1}{\rho_2} \end{pmatrix} \mathcal{R}^{-1} \tag{68}$$

$$X S_4 X^T = \begin{pmatrix} \frac{1}{m\rho_1^2} & 0 \\ 0 & \frac{1}{m\rho_2^2} \end{pmatrix} \tag{69}$$

$$(1 + \mathrm{i}B)C^{-1} = -\mathrm{i}m\mathcal{R} \begin{pmatrix} \frac{\dot{\rho}_1}{\rho_1} + \mathrm{i}\frac{1}{m\rho_1^2} & 0 \\ 0 & \frac{\dot{\rho}_2}{\rho_2} + \mathrm{i}\frac{1}{m\rho_2^2} \end{pmatrix} \mathcal{R}^T. \tag{70}$$

The wavefunction is then written as

$$\Psi_{n_1,n_2}(x_1, x_2, t) = \left[\frac{1}{\pi n_1! n_2! 2^{n_1} 2^{n_2} \rho_1 \rho_2}\right]^{1/2} \mathrm{e}^{-\mathrm{i}\int^t \mathrm{d}t' [\frac{1}{m(t')\rho_1(t')}(n_1+\frac{1}{2}) + \frac{1}{m(t')\rho_2(t')}(n_2+\frac{1}{2})]}$$
$$\times \mathrm{e}^{-\frac{1}{2}[-\mathrm{i}m(t)\frac{\dot{\rho}_1}{\rho_1} + \frac{1}{\rho_1^2}]\xi_1^2 - \frac{1}{2}[-\mathrm{i}m(t)\frac{\dot{\rho}_2}{\rho_2} + \frac{1}{\rho_2^2}]\xi_2^2} H_{n_1}\left(\frac{\xi_1}{\rho_1}\right) H_{n_2}\left(\frac{\xi_2}{\rho_2}\right) \tag{71}$$

where $(\xi_1, \xi_2)^T = \mathcal{R}^T \boldsymbol{q}$. In the case of $\lambda(t) = 0$, we can see that the wavefunction (71) becomes the product of those one-dimensional time-dependent harmonic oscillators.

5. Summary

In summary, for N-coupled time-dependent harmonic oscillators, we have obtained the primary invariant and LR-type invariant in terms of classical solution matrix by using the Heisenberg picture approach. Using the LR-type invariant and exploiting the properties of symplectic matrix, we have calculated the eigenfunctions of the LR-type invariant. The exact wavefunctions of the Schrödinger equation were also calculated using the LR invariant method and unitary transformation. We also gave simple one- and two-dimensional examples. For the one-dimensional case, it is found that the wavefunctions coincide with the previous results [6, 7, 9].

The time evolution operator, coherent state, squeezed state, propagator, Berry phase, etc, for such a system will be presented elsewhere. Our method can also be easily applied to the system having driving force terms. We hope that our result gives new light on the physics related to this issue.

Appendix

We are concerned here with derivation of relation (29) and related examples. First, it is noted that $\mathcal{A} = W^{-T}\Gamma^T\Gamma W^{-1} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ is a symplectic matrix, i.e. it satisfies $\left(W^{-T}\Gamma^T\Gamma W^{-1}\right)^T J \left(W^{-T}\Gamma^T\Gamma W^{-1}\right) = J$ [23]. Thus from

$$\mathcal{A}^T J\mathcal{A} = \begin{pmatrix} AB^T - BA & AC - B^2 \\ B^{2T} - CA & B^TC - CB \end{pmatrix} = J \tag{72}$$

it follows that

$$AB^T = BA \qquad AC - B^2 = I_N \qquad B^TC = CB \tag{73}$$

where we have used that $\mathcal{A}$ is a symmetry matrix. Since the matrix C is nonsingular, we obtain that

$$C^{-1}B^T = BC^{-1} \tag{74}$$

and

$$A - BC^{-1}B^T = C^{-1}. \tag{75}$$

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